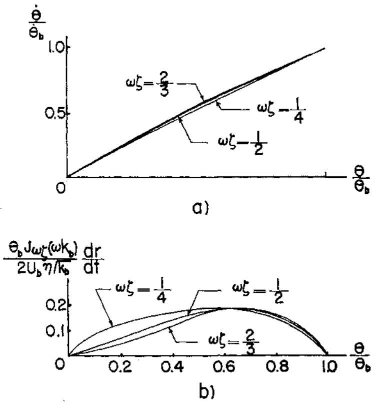

Fig. 3 Angular and radial velocities as functions of angle-to-go.

with respect to the planet are known. From Eq. (23), one has

$$\frac{\theta_b J_{\omega\zeta}(\omega k_b)}{2\omega k_b U_b} W_b = \frac{\omega\zeta}{\omega k_b} J_{\omega\zeta}(\omega k_b) - J_{\omega\zeta + 1}(\omega k_b) \tag{26}$$

This equation may be solved for its characteristic root

$$\omega k_b = \eta \tag{27}$$

provided that θ_b, U_b, W_b, and $\omega\zeta$ are given. It is emphasized here that η is the lowest positive real root of Eq. (26).

An example is given here for descent from a circular orbit ($W_b = 0$) along a trajectory of Bessel function of $\frac{1}{2}$ order. Thus, from Eq. (23), one obtains

$$\frac{\theta_b \sin(\omega k_b)}{U_b \omega k_b}\left(\frac{dU}{d\theta}\right)\bigg|_{\theta=\theta_b} = 2\cos\left(\omega k_b \frac{\theta^2}{\theta_b^2}\right) - \left(\omega k_b \frac{\theta^2}{\theta_b^2}\right)^{-1} \sin\left(\omega k_b \frac{\theta^2}{\theta_b^2}\right)\bigg|_{\theta=\theta_b} = 0 \tag{28}$$

from which one has

$$\tan\omega k_b = 2\omega k_b \tag{29}$$

The value $\omega k_b = \eta = 1.16562$ may be obtained for its fundamental root for the case $\omega\zeta = \frac{1}{2}$.

The dimensionless form of $dr/d\theta$ is shown in Fig. 2b for $\omega k_b = \eta$ and various values of $\omega\zeta$.

5. Angular and Radial Velocities

With the aid of Eqs. (4) and (11), the angular velocity $\dot\theta$ may be written as

$$\dot\theta/\dot\theta_b = (\theta/\theta_b)(u^2/u_b^2) \tag{30}$$

From Eqs. (18, 19, and 30), one has

$$\frac{\dot\theta}{\dot\theta_b} = \left(\frac{u_0}{u_b}\right)^2 \left(\frac{\theta}{\theta_b}\right) \left[1 - \frac{U_b}{u_0} \frac{J_{\omega\zeta}[\omega k_b(\theta^2/\theta_b^2)]}{J_{\omega\zeta}(\omega k_b)}\right]^2 \tag{31}$$

The value of $\dot\theta$ at $\theta = 0$ is zero, as shown in Fig. 3a. This is in conformity with the requirement of soft landing. Also,

$$dr/dt = u^{-2}(dU/dt) \tag{32}$$

where

$$\frac{dU}{dt} = \frac{d\theta}{dt}\frac{dU}{d\theta} = k^{1/2}u^2 \frac{dU}{d\theta} \tag{33}$$

Combining Eqs. (11, 23, 32, and 33), one obtains

$$\frac{\theta_b J_{\omega\zeta}(\omega k_b)}{2\omega k_b U_b}\frac{dr}{dt} = \left(\frac{\theta}{\theta_b}\right)^2 \left[\frac{\zeta}{k_b}\left(\frac{\theta}{\theta_b}\right)^{-2} J_{\omega\zeta}\left(\omega k_b \frac{\theta^2}{\theta_b^2}\right) - J_{\omega\zeta + 1}\left(\omega k_b \frac{\theta^2}{\theta_b^2}\right)\right] \tag{34}$$

which also is plotted in Fig. 3b for $\omega k_b = \eta$ and various values of $\omega\zeta$.

It is expected that the reasoning given in discussing Eq. (24) holds also here. Thus, the order of magnitude of the term dr/dt is

$$\theta^{2 - 2 + 2\omega\zeta} = \theta^{2\omega\zeta} \tag{35}$$

Therefore,

$$dr/dt|_{\theta=0} = 0 \qquad \text{for } 0 < \omega\zeta \tag{36}$$

Equation (36) indicates that the radial velocity of the vehicle is also zero at the point of contact on the surface of a planet, as would be expected.

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Thermodynamic Calculation of Partly Frozen Flow

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In some analyses of expanding flow with finite chemical kinetics, it is desirable to interpret performance in terms of more than one freezing point. For example, in the expansion of highly dissociated air from a shock tunnel, exact thermokinetic calculations demonstrate that nitric oxide will effectively freeze earlier than atomic oxygen and atomic nitrogen. Thus, in approximate thermodynamic calculations employing fixed assignments of freezing points, it is necessary at some point to be able to provide for nonreactive nitric oxide, i.e., nitric oxide that cannot disappear thermodynamically upon further cooling in compliance with the chemical equilibrium of the reaction $2\,NO = N_2 + O_2$. However, most generalized computer programs involving chemical equilibrium among a mixture of elements necessarily are designed to insure that all possible, individual equilibrium relations are satisfied. If nitric oxide, molecular oxygen, and molecular nitrogen are

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provided for as possible species, the generalized computer program will *insist* that the chemical equilibrium between them be satisfied unless *all* compositions are frozen. One can, usually, leave out nitric oxide (for example) entirely, but this is no answer to the problem of freezing it alone.

A means of performing thermodynamic calculations for partially frozen flows has been devised which is more or less exact, depending upon the particular circumstances. This involves creating, algebraically, a whole new periodic table of the elements and resultant chemical system. This chemical system "beyond the looking glass" is uniquely designated as the Q system, consisting of QH, QHe, QLi, etc. All members of the Q chemical system, of course, have identically the same properties as their normal counterparts (molecular weight, enthalpy, entropy, heats of formation), but only physically, not chemically, can the two systems coexist. The generalized computer program can react both $2\ QNQO = QN_2 + QO_2$ and $2\ NO = N_2 + O_2$, but neither $2\ QNO = QN_2 + O_2$ nor $2\ NQO = N_2 + QO_2$.

Now it is obvious that, if the existence of QN_2 and QO_2 is forbidden to the program by leaving out these species, none of the $QNQO$ nitric oxide can participate in the composition shift to these molecules called for by equilibrium. If the existence of NO nitric oxide is forbidden on the "proper side of the looking glass," no N_2 and O_2 can react to fill an algebraic void. In certain generalized programs, the equilibrium existence of the atomic species *must* be allowed for, since the gaseous atoms are the basic building blocks of the entire chemical system. Thus, the equilibrium adjustment of the reaction $QNQO = QN + QO$ still must be permitted. For this particular reaction, this involves the existence of a trivial amount of atomic nitrogen and atomic oxygen which thermodynamically ought not to be distinguished from their non-Q counterparts. Thus, it must be stated that in such programs the approach is only more or less exact; other programs avoiding the inherent necessity of the presence of the atoms as building blocks can apply the approach exactly. By definition, if no more of the frozen species is to be created or consumed, it need appear only on the Q side of the looking glass. Thus, one can avoid the anomaly of computing an entropy level for a mixture where both the Q species and its normal counterpart contribute separately to the entropy of mixing.

In application of this approach to the dissociated air example, a normal equilibrium expansion is carried out until the freezing of nitric oxide is indicated. At this point, equilibrium calls for the reaction $QN + QO = QNQO$ to lie very preponderantly to the right. Thus, virtually (but not *exactly*) no error is introduced by assigning all of the NO, and nothing else out of the present equilibrium mixture, as input $QNQO$ on a mass balance basis. From the same pressure, temperature, and (necessarily) enthalpy level, the expansion now is continued. All of the original species except NO are permitted in the non-Q system; only $QNQO$, QN, and QO are permitted in the Q system. Thus, atomic oxygen and atomic nitrogen may continue to recombine to their respective molecular states as the individual equilibrium relationships dictate. When these recombination reactions are indicated to freeze, the normal program option of total freezing can be invoked.

The Q approach has been used to attain thermal equilibrium, without chemical reaction, of hydrogen fuel in hot air. The QH_2 is, of course, allowed to undergo its own equilibrium dissociation: $QH_2 = 2\ QH$. The same sort of treatment could, of course, be applied to equilibrium pyrolysis of hydrocarbon fuel in a heated air stream, even one already containing some equilibrium combustion products. By neglecting the entropy-of-mixing problem alluded to earlier, an application of the Q approach to the mixing of two reactive streams of fuel and air of different stoichiometries is envisioned, in order to separate thermal and molecular weight effects from chemical effects.

Comments

Further Comment on "Fundamentals of Boundary-Layer Heat Transfer with Streamwise Temperature Variation"

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COMMENTS by Tribus[5] on a paper of Biot[1] were critical of Mr. Biot for having omitted reference to a body of articles previously written in the field. In defense of his position, Biot indicated that these references were permeated with errors and chose to support this point by quoting an alleged conclusion from one of my reports.[2] This requires two comments. First, the quotation cited by Biot does not appear anywhere in the conclusions of that report. Second, the alleged conclusion is not very damaging, in that it merely states that a plate having a variable surface temperature experiences local convective heating, which even Ref. 1 shows. I can only surmise, from these two bits of evidence, that Biot was careful neither in reading my report nor in writing his reply. Certainly, he did not misquote me intentionally, but perhaps he meant only to paraphrase my conclusions and had forgotten to insert the term "coefficient" after "heat transfer." But, this isn't a small point. It reveals that Biot has fallen into the same trap as I did in 1947,[3,†] when I also chose to abuse the concept of the heat transfer coefficient under conditions of variable surface temperature. After reconsidering the problem, I intended to show in Ref. 2 that it is not really the heat transfer coefficient that is at fault in the Newton Law of Cooling expression, but that the problems arise from the use of a local temperature potential made up of the difference of the local wall and freestream temperatures. A proper temperature potential replaces the local surface temperature with a term that reflects the temperature variations along the plate from the start of the boundary layer in the form of a Duhamel integral. When this is done, the out-of-phase anomalies rediscovered in Ref. 1 no longer occur. The heat transfer coefficient becomes merely a measure of the flow conditions and the state and transport properties of the medium. The basis of this same conclusion emerges so vividly from Biot's own variational approaches to the problem[4] that one wonders who really is replacing understanding with elaborate analytical procedures.

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[†] Unfortunately without being aware of Leveque's pioneering efforts in this field.